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RATIONAL MODELING BY

PENCIL-OF-FUNCTION METHOD

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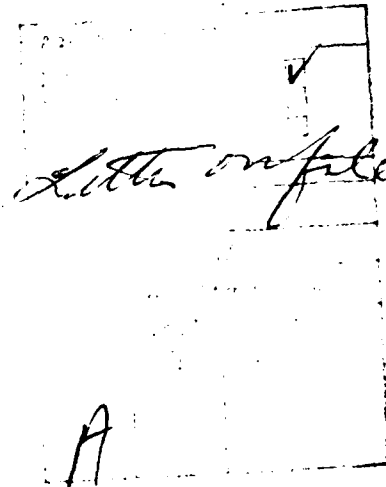1 JAN 73
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RATIONAL MODELING BY PENCIL-OF-FUNCTIONS METHOD

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ABSTRACT

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I. INTRODUCTION

Signal representation and approximation [1]-[4] is basic to (a) time-domain extraction of singularities of a scatterer's field pattern [5],[6],[16] and to (b) recursive digital-filter synthesis [7], [12]. It is also useful in (c) bandwidth

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compression of signals [2], and (d) time-domain measurement and testing of networks/channels. This paper discusses a unified approach to representing or approximating a given empirical signal x(t) by sum of exponentials, i.e., for finding the right hand side of

$$x(t) \cong y(t) = \sum_{i=1}^{n} d_i \, e^{s_i t} \leftrightarrow Y(s) \qquad (1a)$$

$$Y(s) = \sum_{i=1}^{n} \frac{d_i}{s - s_i} \qquad (1b)$$

or, equivalently, the right hand side of the sampled version

$$x(k) \cong y(k) = \sum_{i=1}^{n} R_i (z_i)^k \leftrightarrow Y(z) \qquad (2a)$$

$$Y(z) = \sum_{i=1}^{n} \frac{R_i}{(1 - z_i z^{-1})}$$

$$= \frac{b_0 + b_1 z^{-1} + \ldots + b_{n-1} z^{-n+1}}{1 + a_1 z^{-1} + \ldots + a_n z^{-n}}$$

$$= \frac{B(z)}{A(z)} \qquad (2b)$$

The poles s_i (or z_i in z-domain) are either real, or they occur in complex conjugate pairs.

When equality holds in (2a), the sampled signal $x(k)$ is said to be rational of order n, and thus rationally representable. Additionally, if Re $s_i < 0$ (or, $|z_i| < 1$) it is said to be stable-rational of order n.

In the method described here, the given signal is processed in reverse-time by a cascade of first order digital filters to yield a family of information signals. The Gram matrix F of these information signals is shown to contain the essential information on the denominator parameters of $Y(z)$.

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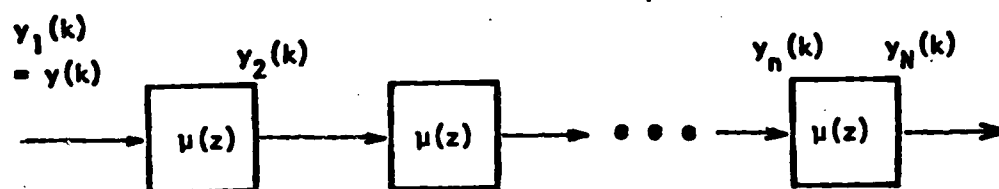
$$\mu(z) = 1/(1 - qz)$$

Fig. 1. Reverse-time processing by
first-order filters

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Specifically, it is shown that A(z) is determined as

$$A(z) = (qz)^{-n} \left[\sum_{i=1}^{n+1} \sqrt{D_i} (qz-1)^{n+1-i} \right] / \sqrt{D_1}$$

where D_i are the diagonal cofactors of the matrix F. The numerator parameters are then determined using a least-squares fit, i.e., $\underline{b} = -P^{-1}\underline{c}$, where P and $\underline{c}$ are defined in the paper.

The entire procedure is thus noniterative and computationally efficient. Iterative methods, such as the modified Newton method [12], require as many as a hundred iterations, each involving a matrix inversion. Our computations are roughly equivalent to two matrix inversions. It is a generalization of the method developed in [8] to reverse-time processing by first order filters. This formulation results in a lower order matrix (n+1 dimensional) than did the formulation in [8] (2n+1 dimensional). Examples presented demonstrate (i) noiseworthiness in the representation problem when data are corrupted by noise and (ii) the effectiveness of the method in the approximation problem. Comparison of the method with the maximum entropy method (or all-pole linear predictor) and the Prony method [1], [4] is also included in the paper.

II. FIRST-ORDER FILTER BASED INFORMATION SIGNALS

Suppose a suitable K has been selected such that $x(k) = 0$ for $k \geq K$ (so that use of the upper limit K-1 instead of ∞ on summations may be permitted). We define the <u>reverse-time</u> first-order filtered signals as (see Fig. 1)

$$x_1(k) = x(k)$$

$$x_2(k) = qx_2(k+1) + x_1(k) \qquad (3)$$

$$\cdot$$

$$\cdot$$

$$x_N(k) = qx_N(k+1) + x_n(k)$$

4

where $N = n+1$, and $x_i(K) = 0$ for $i = 1, 2, \ldots, N$. Further, $0 < q < 1$.

This family of signals[1], which we shall call information signals, possesses the interesting property stated below.

Lemma 1

If $x(k) = y(k)$ is stable-rational of order n with poles z_i, then the corresponding information signals are also stable rational of order n with the poles z_i:

$$y_{i+1}(k) = \sum_{\ell=1}^{n} \frac{R_\ell}{(1-q\,z_\ell)^i} (z_\ell)^k \qquad (4)$$

Proof: We prove this by induction. For $i=0$ the statement is trivially true since (4) is identical to (2) for this case. Assuming it to be true for $i-1$, let us proceed to prove it is true for i.

From (3)

$$y_{i+1}(k) = qy_{i+1}(k+1) + y_i(k)$$

which is readily shown to be equivalent to

$$y_{i+1}(k) = \sum_{\nu=k}^{\infty} q^{\nu-k} \, y_i(\nu)$$

$$= \sum_{\ell=1}^{n} \frac{R_i}{(1-q\,z_\ell)^{i-1}} \sum_{\nu=k}^{\infty} q^{\nu-k}(z_\ell)^\nu$$

$$\text{(from induction hypothesis)}$$

$$= \sum_{\ell=1}^{n} \frac{R_i}{(1-q\,z_\ell)^{i-1}} z_\ell^k \sum_{\nu=k}^{\infty} q^{\nu-k}(z_\ell)^{\nu-k}$$

The result of equation (3) follows immediately by observing that

---

[1] The nomenclature 'information signal' is not to be confused with traditional information theoretic concepts. It is used here because these signals will be shown to yield the denominator parameters of $Y(z)$.

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$$\sum_{\nu=k}^{\infty} q^{\nu-k} (z_\ell)^{\nu-k} = \frac{1}{(1-q\, z_\ell)}$$

Before leaving this section, we remark that the set $y_1, \ldots, y_n$ is linearly independent, while the set $y_1, \ldots, y_n, y_{n+1}$ is linearly dependent.

III. DETERMINATION OF PARAMETERS VIA PENCIL-OF-FUNCTIONS THEOREM FOR RATIONALLY REPRESENTABLE SIGNALS

In this section we will determine the signal parameters for the case $x(k) = y(k)$, i.e., where the signal is rationally representable. We will call z_ℓ (see (2)) the poles of the impulse response, R_ℓ the corresponding residues, and $\zeta_\ell = \{(z_\ell)^k\}$ the associated modes. Note that the poles occur in conjugate pairs whenever complex, as do the residues, since y is real.

The significance of Lemma 1 of the previous section arises from the fact that each of the information signals contains the modes $\zeta_\ell = \{(z_\ell)^k\}$, $\ell = 1, \ldots, n$. Further, the pencil-of-signals[2] $\gamma\, y_{i+1} + y_i$ also contains all these modes unless γ equals one of the poles; in the latter case, i.e., when $\gamma = z_m$, $\gamma\, y_{i+1} + y_i$ does not contain the mode $\zeta_m = \{(z_m)^k\}$. This results in the following observation

Lemma 2. The set

$$(qz_m - 1)y_2 + y_1, \quad (qz_m - 1)y_3 + y_2, \quad \ldots, \quad (qz_m - 1)y_N + y_n \qquad (5)$$

is linearly dependent for $m = 1, 2, \ldots, n$ where z_m are the poles of the right hand side of (2).

Definition. Define the $N \times N$ dimensional Gram matrix (recall, $N = n+1$)[11]

---

[2] The terminology pencil-of-functions is derived from literature in physics and mathematics; see, for example, Gantmacher [9] where $A + \lambda B$ is called a pencil of matrices A and B parametrized by sealer λ. See also Gueguen[10] for recent usage.

$$F = \begin{bmatrix} \langle y_1, y_1 \rangle & \cdots & \langle y_1, y_N \rangle \\ \cdot & \cdots & \cdot \\ \cdot & \cdots & \cdot \\ \cdot & \cdots & \cdot \\ \langle y_N, y_1 \rangle & \cdots & \langle y_N, y_N \rangle \end{bmatrix},$$

$$\langle y_i, y_j \rangle = \sum_{k=0}^{K-1} y_i(k) y_j(k) \tag{6}$$

or, equivalently

$$F = \sum_{k=0}^{K-1} \underline{f}(k) \underline{f}^T(k) \tag{7}$$

where

$$\underline{f}^T(k) = [y_1(k)\ y_2(k)\ \cdots\ y_N(k)].$$

We can now apply the pencil-of-functions theorem of reference [8] to obtain the central theoretical result of this section. A statement of pencil-of-functions theorem is given in Appendix A.

Theorem 1. The poles of the impulse response y(k) must satisfy the equation

$$\sum_{i=1}^{N} \sqrt{D_i}\ (qz-1)^{N-i} = 0 \tag{8}$$

where D_i are the diagonal cofactors of the Gram matrix F (defined in (7)). Proof: The theorem follows immediately upon application of the pencil-of-functions theorem (reference [8]) to the set (5).

Note that the denominator of transform of the impulse response is given by

$$A(z) = D_1^{-1/2}(qz)^{-n} \sum_{i=1}^{N} \sqrt{D_i}\ (qz-1)^{N-i} \tag{9}$$

This follows from (8) by dividing through by z^n and by normalizing the coefficients so that the leading coefficient becomes unity.

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The numerator parameters can be found by the method of least squares, specifically by solving the linear equation

$$P \underline{b} = \underline{c} \tag{10}$$

where $\underline{b} = [b_o \; b_1 \; \ldots \; b_{n-1}]^T$, $\underline{c} = [c_o \; c_1 \; \ldots \; c_{n-1}]^T$, $P = [p_{ij}]$; and

$$\bar{p}_{ij} = <u_i, \; u_j> \tag{11a}$$

$$c_i = <u_i, \; y> \tag{11b}$$

Here u_i denotes the impulse response of $z^{-1}/A(z)$. Note that $u_i(k) = u(k-i)$ where $u(k)$ is the impulse response (i.e., inverse z-transform) of $1/A(z)$. All inner products are summed from $k = 0$ to $K-1$.

Remarks. Before leaving this section we remark that the parameters characterizing the signal, i.e., the coefficients of the polynomials $A(z)$ and $B(z)$, are recovered exactly. It is assumed of course that the signal is of the form (1) and that the true model order is known.

The idea of reverse-time integration was proposed by Carr in [13] and Jain in [14]. Here, we have generalized the concept of reverse-time processing to the case of first-order filter processing. Note that the first order filter $1/(1-qz)$, used above, encompasses integration; just let $q = 1$.

IV. MODELING IN THE PRESENCE OF NOISE

In the last section we modeled an ideal signal from its samples. The effect of additive noise is now considered. The samples available are

$$x(k) = y(k) + \sigma \, w(k) \tag{12}$$

where $w(k)$ is a zero mean white noise process and σ is an unknown positive con-stant. As in Section II, the information signals are again generated by processing $x(k) = x_1(k)$ by the filter cascade of Fig. 1. (see Fig. 2).

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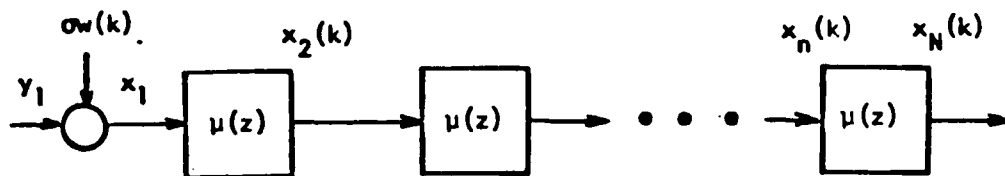
Fig. 2. Noisy signal through first-order filters

Note, because of the linearity of the filters, that

$$x_1(k) = y_1(k) + \sigma w_1(k) \qquad (13)$$

Then it can be seen that the expected value of the Gram matrix G of the vector signal $\underline{x}(k) = [x_1(k) \; x_2(k) \; \ldots \; x_N(k)]^T$ is[3]

$$E\,G = E \sum_{k=0}^{K-1} \underline{x}(k) \; \underline{x}^T(k) = F + \sigma^2\,W \qquad (14)$$

where W is the covariance matrix of the unit noise vector sequence $\underline{w}(k) = [w_1(k) \; w_2(k) \; \ldots \; w_N(k)]^T$ and is known before hand. To estimate σ^2 and F we use the following criterion.

Jain's Identification Criterion [15]

Consistent with the noise and bias models the estimated Gram matrix should achieve a minimum possible determinant.

Using the above criterion and equation (14), the following estimation procedure has been developed.

Step 1. Estimate σ^2 as

$$\hat{\sigma}^2 = \frac{1}{G^{-1} \odot W} \qquad (15)$$

where $\odot$ denotes matrix inner product (i.e., $Z \odot W = \Sigma\Sigma \; z_{ij} \; w_{ij}$).

Step 2. $\hat{F} = G - \hat{\sigma}^2\,W \qquad (16)$

Step 3. Use $\hat{F}$ in estimating A(z) and B(z) via (9) and (10).

The justification for formulas (15) and (16) is given in Appendix B.

---

[3]The expected value operator is denoted as E

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True impulse response

$$y(k) \longleftrightarrow \frac{1 - 1.92z^{-1} + z^{-2}}{1 - 2.68z^{-1} + 2.476z^{-2} + 0.782z^{-3}}$$
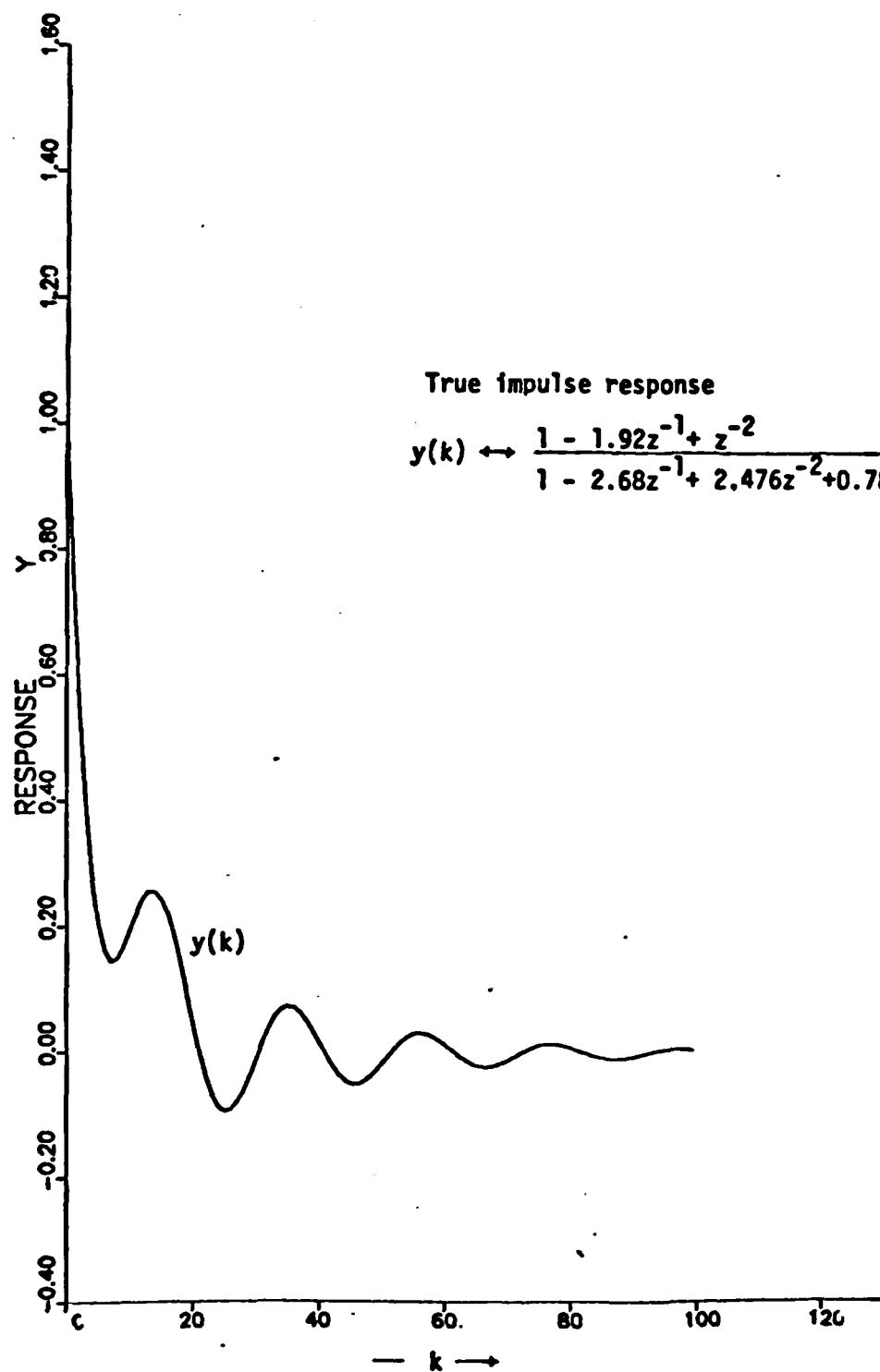
Fig. 3(a) True impulse response of a third order transfer function

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SIGNAL UNDER TEST

= true signal + noise

= $y(k) + \sigma w(k)$

SNR = 12 dB

$x(k)$
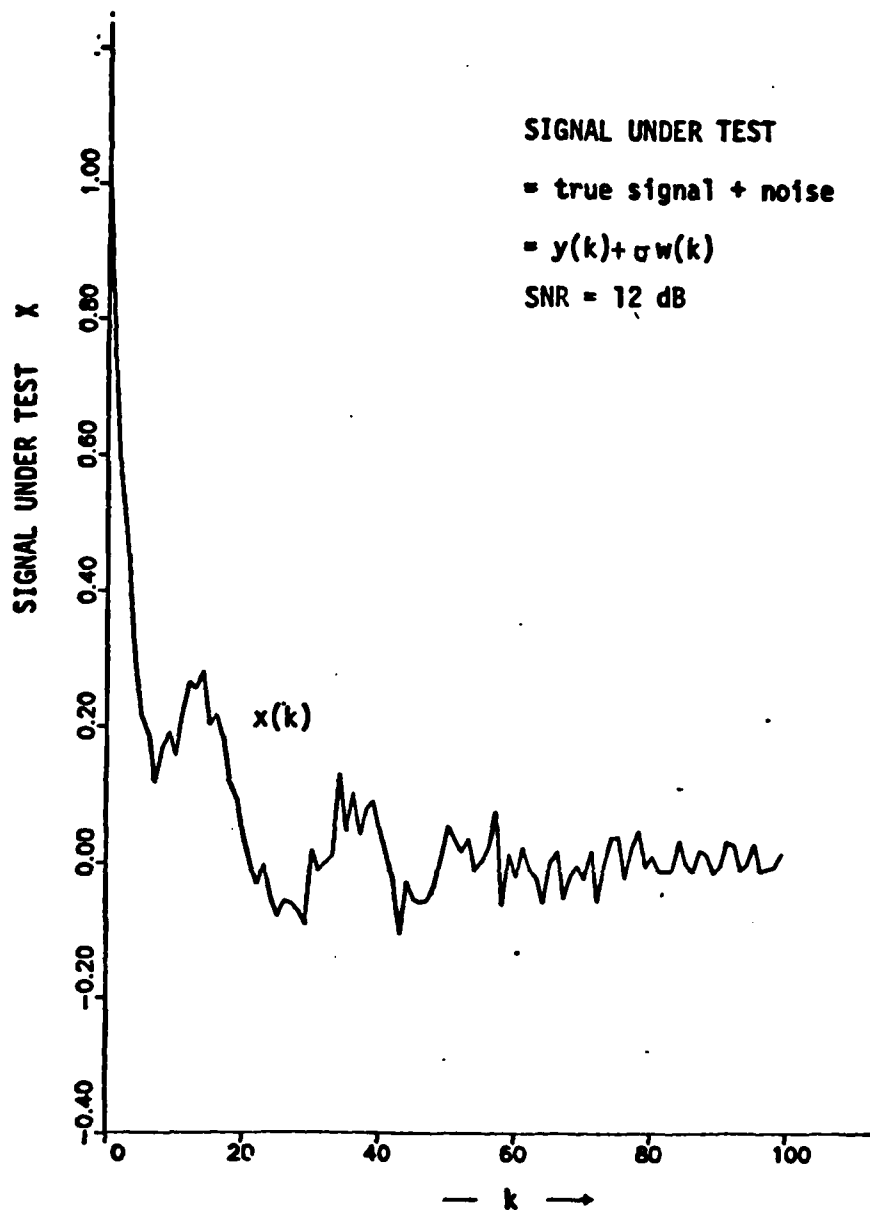
Fig.3(b) A simulated noisy signal under test

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V. A COMPUTER SIMULATION EXAMPLE

Let

$$y(k) \leftrightarrow \frac{1 - 1.92z^{-1} + z^{-2}}{1 - 2.68z^{-1} + 2.476z^{-2} - 0.782z^{-3}}$$

$$= \frac{(1 - e^{j\beta}z^{-1})(1 - e^{-j\beta}z^{-1})}{(1 - 0.8452z^{-1})(1 - re^{j\theta}z^{-1})(1 - re^{-j\theta}z^{-1})} \qquad (17)$$

$$(\beta = 0.28379; \ r = 0.96187, \ \theta = 0.30528)$$

be truncated at $k = 99$. The signal to be tested is formed as

$$x(k) = y(k) + \sigma w(k) \qquad (18)$$

where $w(k)$ is a zero mean, uncorrelated noise sequence. The positive scalar σ is chosen to be 0.0425 so that the signal-to-noise ratio is 12 dB. The true signal $y(k)$ and the noisy signal $x(k)$ are shown in Fig. 3.

The signal under test was modeled by

1. Pencil of functions method (without applying noise correction); reverse-time processing pole q was taken as 0.8.

2. Pencil-of-functions method with estimation of σ^2 and F; the value of q was again taken to be 0.8.

3. All-pole covariance[4] technique. Minimum error criterion, rather than equal energy criterion was used to establish the gain parameter.

4. Pole-zero model using Prony method. Note that the denominator parameters (and, of course, the poles) are the same as that for the all-pole covariance method [2]. The numerator parameters are then determined by least-squares fit.

Fifty simulation runs, each with a different sample of noise, were performed. The 'mean' and 'standard deviation (S.D.)' of the various quantities

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[4] The all-pole autocorrelation method yields very similar results in the present case; hence, the autocorrelation method is not included.

of interest are shown in Table 1. Of course, the model order was taken to be 3 and the number of signal samples used were 100.

The poles for the above fifty runs are listed in Appendix C, Tables C1-C3. A graphical portrayal of the z-domain poles for 10 runs is given in Figures 4 to 6. The location of the true poles is at the centers of the circles shown in these figures. Judging from the normalized mean square errors in Table 1, as well as from the scattergrams of the poles, it appears that the pencil-of-functions method can perform reliable modeling of a rational signal even when it is masked by noise. As is widely known, the Prony method (and of course the covariance method) perform quite poorly in the presence of noise.

It is sometimes claimed that the Prony method (or the all-pole-covariance method) performs well with short data-frame. We give the poles of ten runs with the first 10 data points of the noisy signal used in the above experiments. These poles are given in Fig. 7. Again a wide and unreliable scatter of the poles is produced.

VI. APPLICATION TO AN ELECTROMAGNETIC PULSE (EMP)

As a real world application we consider the use of pencil-of-functions method to the transient response of a conducting pipe tested at the ATHAMAS-I EMP simulator. The conducting pipe is 10m long and 1m in diameter. Hence, the true resonance of the pipe is expected to be in the neighborhood of 14MHz. Also, the pipe has been excited in such a way that it is reasonable to expect only odd harmonics at the scattered fields. The data measured are the integral of the E-field: i.e., the measured variable is a voltage. The transient response used for analysis is shown in Fig. 8 by the solid line. The results of analysis by the pencil-of-functions method are given in Table 2 for an 8th order model; the model response, with an error of 0.0125, is shown in Fig. 8

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Table 1

Comparison of Pencil-of-Functions Method With

All-pole covariance and Prony Methods

(Results of fifty noisy runs: SNR=12 dB)

Mean ± S.D. of	POF	POF with noise correction	All-pole covariance	Prony
a_1	-2.5974 ± 0.0216	-2.6753 ± 0.0185	-0.5200 ± 0.0747	
a_2	2.3438 ± 0.0388	2.4698 ± 0.0331	-0.2334 ± 0.1123	
a_3	-0.7288 ± 0.0187	-0.7803 ± 0.0160	0.0161 ± 0.0710	
Error (NMSE)	0.0314 ± 0.0095	0.0055 ± 0.0052	0.1726 ± 0.0164	0.1499±0.0098

Note: From (17) the true parameters are a_1 = -2.68, a_2 = -2.476 and a_3 = -0.782

The denominator parameters for the all-pole-covariance method are the same as those of the Prony method

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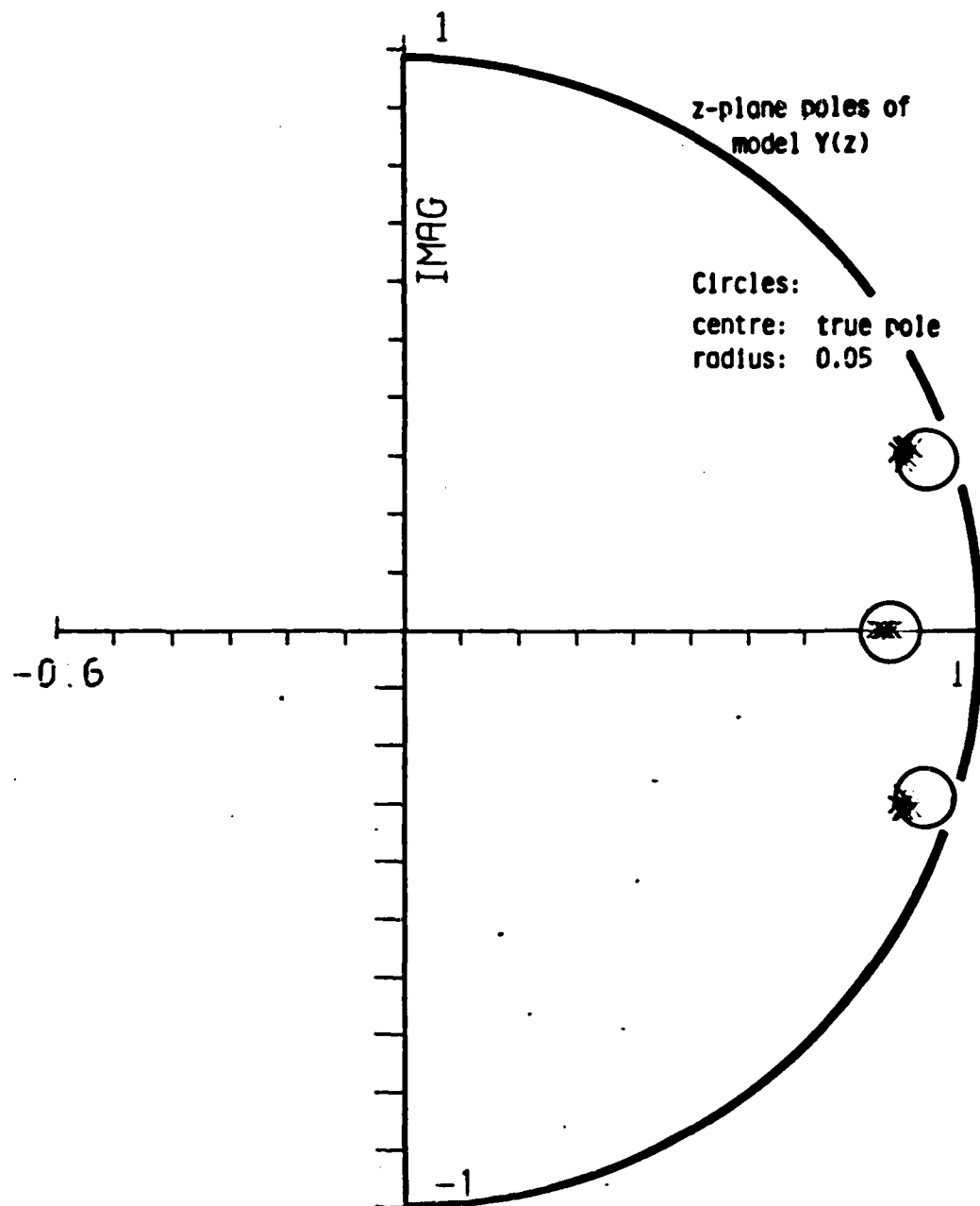
Fig. 4. Poles obtained in ten (10) simulation runs by pencil-of-functions method.

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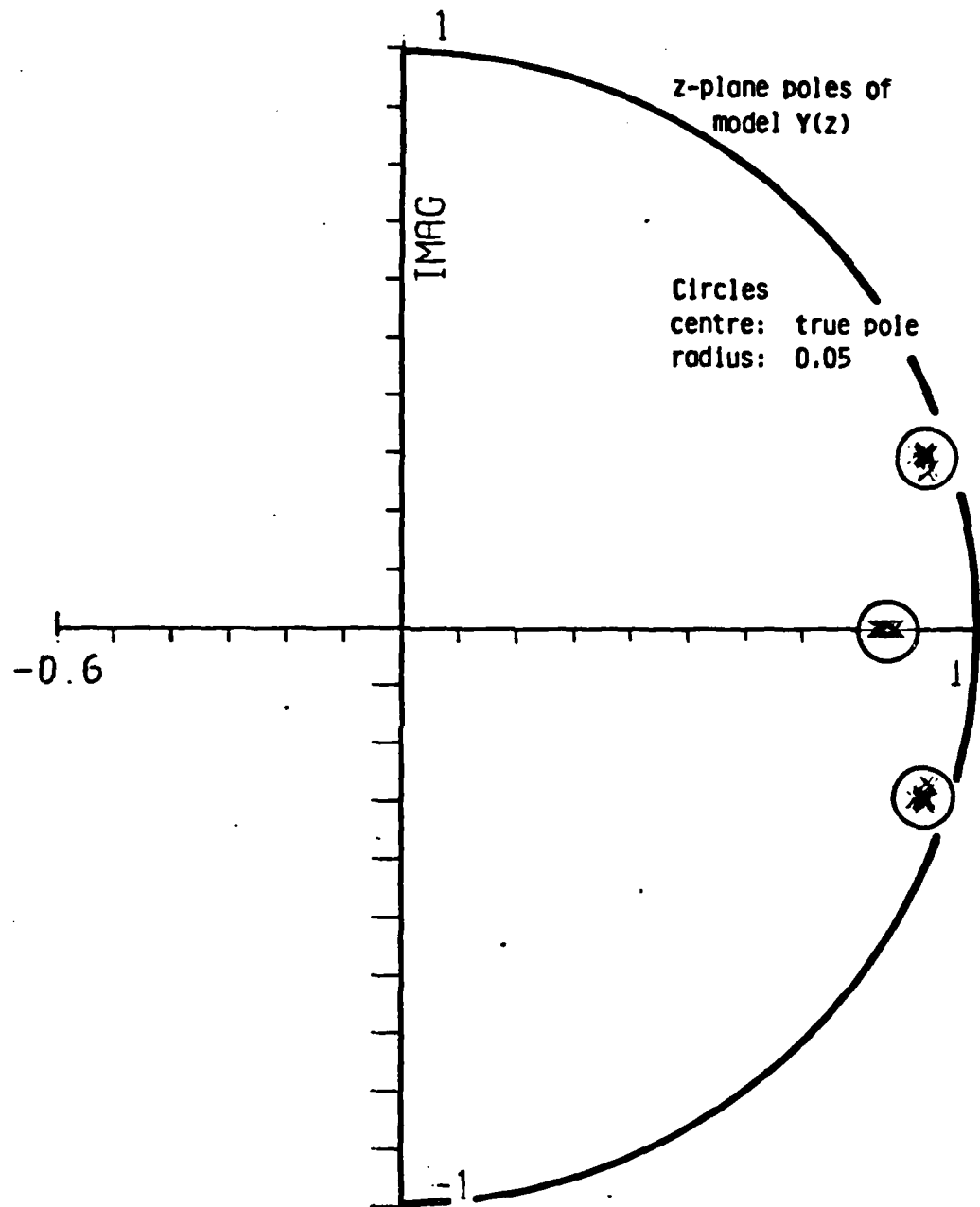
Fig. 5. Poles obtained in ten (10) simulation runs by pencil-of-functions method with noise estimation.

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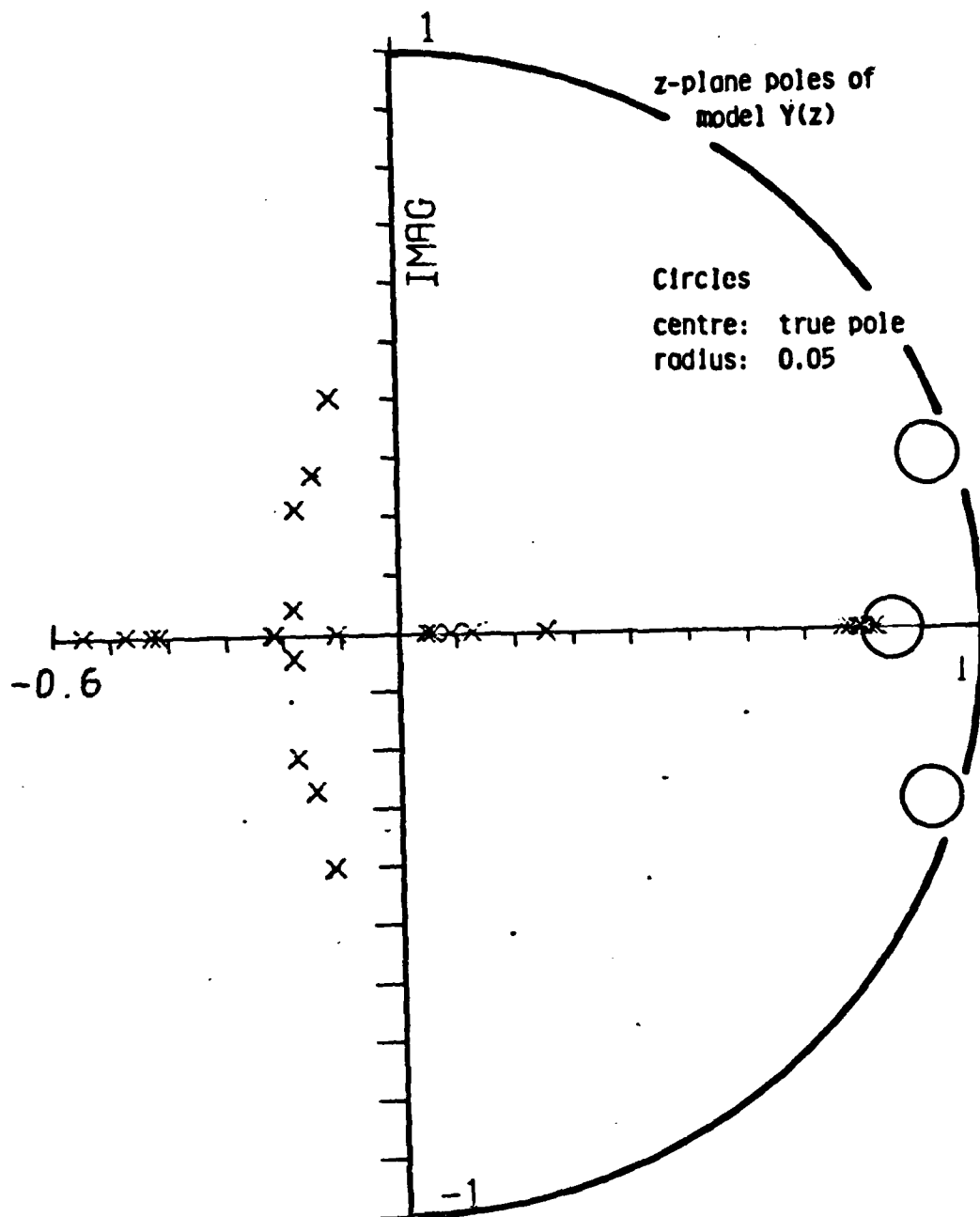
Fig. 6. Poles obtained in ten (10) simulation runs by Prony method.

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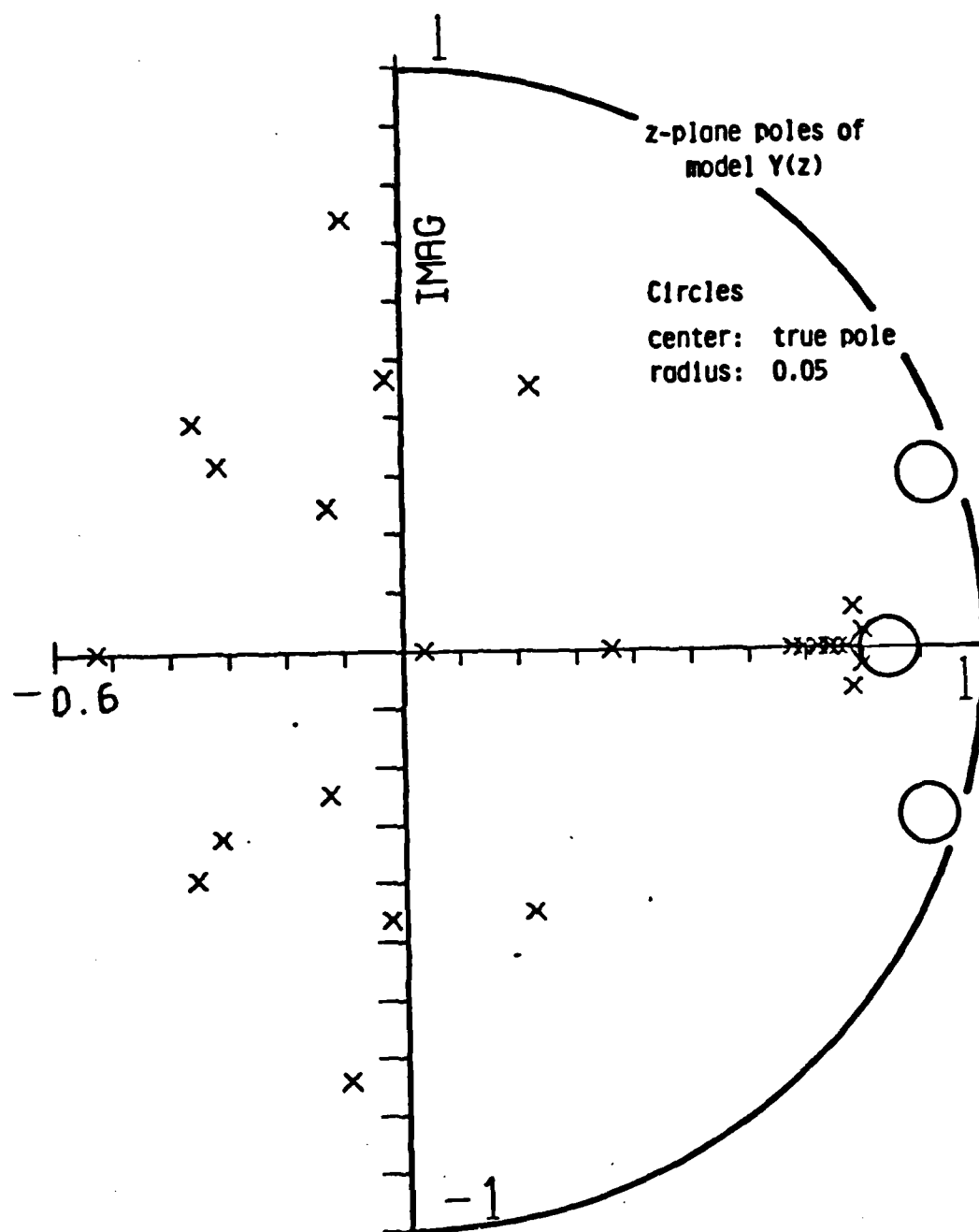
Fig. 7. Poles obtained in ten (10) simulation runs by Prony method with short frame (K=10).

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by the dotted line. The sampling interval is $\Delta = 0.97656$ ns and the number
of samples analyzed is $K = 245$. Note that noise estimation and correction,
as described in Section IV, has been used in the analysis.[4]

Table 2

Poles of a Scatterer Response Estimated by
Pencil-of-Functions Method (q=0.8)

fundamental	$-5.72 \pm j \ 68.63$ Mrad/s	(10.96 MHz)
3rd harmonic	$-30.65 \pm j \ 212.60$ Mrad/s	(33.83 MHz)
curve-fit pair	$-1.95 \pm j \ 8.72$ Mrad/s	(1.42 MHz)
curve-fit pair	$-20.17 + j \ 95.58$ Mrad/s	(15.55 MHz)

VII. RECURSIVE DIGITAL FILTER SYNTHESIS EXAMPLE

As a final example we consider the use of pencil-of-functions technique
to digital filter synthesis. Suppose the desired impulse response is [12]

$$h_d(k) = \begin{cases} 0.25, & k=0 \\ \mathrm{Sin}(0.25k)/k, & 1 \le k \le K = 256 \end{cases}$$

It represents the causal part of the inverse DFT of a low-pass filter with
cutoff at 0.25 Hz. The application of the pencil-of-functions technique,
with $q = 0.4$, yields the following filter

$$H(z) = \frac{0.25 - 0.38841z^{-1} + 0.29346z^{-2} - 0.05783z^{-3} - 0.42025z^{-4}}{1 - 2.46160z^{-1} + 2.78530z^{-2} - 1.52226z^{-3} + 0.34885z^{-4}}$$

with a normalized mean-square error 0.00473. Note that the minimum value
of NMSE, as obtained in [12] by iterative methods,[5] is 0.00346.

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[4]The computer output listings are given in Appendix D.

[5]Seventy iterations were needed to achieve the minimum value.

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CONDUCTING PIPE TEST

—— Measured data

---- Model impulse response

Method: Pencil-of-functions
        with noise estimation
        and correction
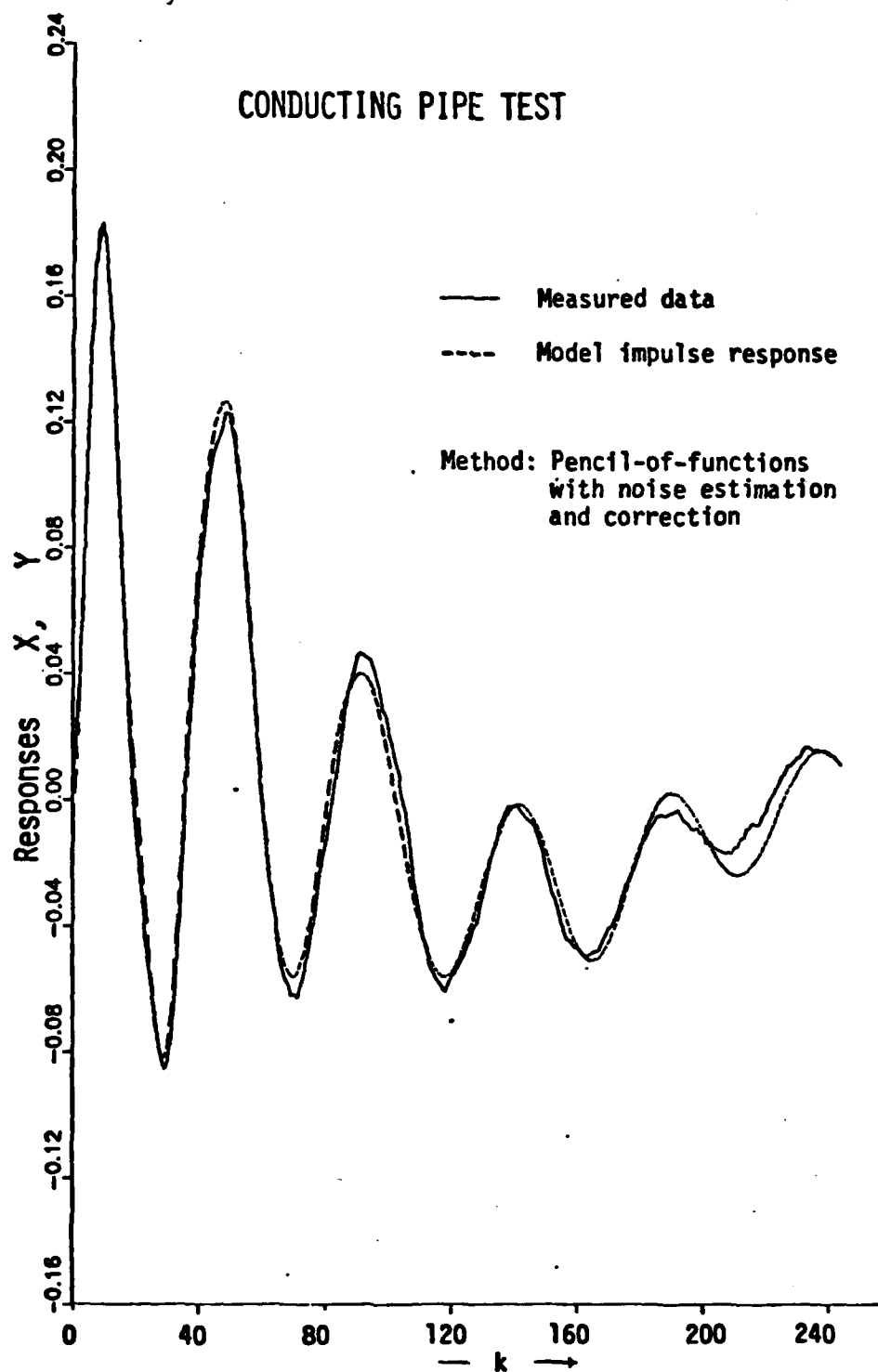
Fig. 8. Comparison of measured data (of the response of
         of a conducting pipe) and model impulse response

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VIII. CONCLUSIONS

Pole-zero modeling of signals has been considered in this paper. It was shown that for rational signals the true parameters can be recovered from the Gram matrix of the information signals. The latter were formed by reverse-time processing of the given signal by a cascade of first order digital filters. Further, we have formulated a new approach to noise estimation and correction by minimizing the determinant of the estimated Gram matrix. The examples demonstrate the practicality of the approach, not only because the computations are noniterative, but also because the poles of the signal are estimated quite accurately. It is felt that the method can be used in a broad range of applications, for example, finding the singularities of a scatter response, modeling of speech and in spectrum analysis.

Extension of the technique to modeling of multichannel signals with common modes (or singularities) is possible. This work is underway. A second area of extension pertains to the case where the filters $\mu(z)$ are chosen to be high-pass. This might be useful when the signal contains an undersirable low frequency drift component.

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APPENDIX A

PENCIL OF FUNCTIONS

A useful mathematical entity arises by combining two given functions defined on a common interval together with a scalar parameter

$$f(t,\gamma) = \gamma g(t) + h(t). \tag{1}$$

We call f a pencil of functions $g(t)$ and $h(t)$ parametrized by γ. To avoid obvious triviality, $g(t)$ is not permitted to be a scalar multiple of $h(t)$.

Our work requires consideration of sets of pencils

$$\gamma g_1(t) + h_1(t), \; \gamma g_2(t) + h_2(t), \; \ldots, \gamma g_n(t) + h_n(t) \tag{2}$$

wherein the functions* $g_1(t),\ldots,g_n(t)$ and $h_1(t),\ldots,h_n(t)$ span separately a common n-dimensional subspace L_n in the function space. For a fixed set of values of parameters γ, the pencils obviously reduce to a set of functions, and the particular values chosen determine properties such as the linear dependence or independence of the set. The main result concerning the linear dependence of pencil sets is derived in [8] and can be stated as follows.

Theorem: Given that the pencil set (2) is linearly dependent, the parameters γ must satisfy the polynomial equation

$$\gamma^n \sqrt{G}[g_1, g_2, \ldots, g_n] \pm \gamma^{n-1} \Sigma \sqrt{G}[g_{i_1}, \ldots; h_{k_1}, \ldots, g_{i_{n-1}}]$$

$$\pm \ldots \pm \gamma \Sigma \sqrt{G}[h_{k_1}, \ldots; g_{i_1}, \ldots, h_{k_{n-1}}]$$

$$\pm \sqrt{G}[h_1, h_2, \ldots, h_n] = 0$$

In every sum term here, the i's and k's form a complete complementary set of indices over the integers $1, 2, \ldots, n$; furthermore, the notation $G[f_1, \ldots, f_n]$ stands for the determinant of the n-dimensional Gram matrix [11]

*All functions are defined on a common interval [a,b], with the usual inner product denoted as

$$\langle f, g \rangle = \int_a^b f(t) g^*(t) dt.$$

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of the functions $f_1, \ldots, f_n$, i.e.,

$$G[f_1, \ldots, f_n] = \det[g_{ik} = \langle f_i, f_k \rangle], \quad i, k = 1, \ldots, n. \tag{5}$$

Lastly, we remark that the sign of each sum term is to be determined as indicated in [8].

The above discussion is equally valid for discrete-time signals. To this end the functions $f(t)$, $g_i(t)$ and $h_i(t)$ must be replaced by the sequences $f(k)$, $g_i(k)$ and $h_i(k)$, and of course the inner product must be redefined as

$$\langle f, g \rangle = \sum_{k=0}^{K-1} f(k) g^*(k)$$

APPENDIX B

NOISE ESTIMATION AND CORRECTION

We observed in Section IV that

$$\mathbb{E}\, G = F + \sigma^2 W \cdot \tag{B1}$$

where G is the Gram matrix of the noisy information signals $x_1, \ldots, x_{n+1}$. Clearly, a good estimator of F is

$$\hat{F} = G - \sigma^2 W \tag{B2}$$

Unfortunately, this estimator is not useable because σ^2 is unknown. We must estimate it using a property of the true information signals states in Section III: The true information signals $y_1, \ldots, y_{n+1}$ are linearly dependent and their Gram matrix is singular. Thus we require

$$|\hat{F}| = |G - \hat{\sigma}^2 W| = 0 \tag{B3}$$

Assuming that $\hat{\sigma}^2$ is small, and retaining only the first two terms of the Taylor series, we have

$$|G| - \hat{\sigma}^2 \Sigma |(G, W)_i| = 0 \tag{B4}$$

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where the notation $(G,W)_i$ stands for the matrix obtained by replacing the ith column of G by the ith column of W.

A little manipulation of (B4) now readily yields relation (16) of Section IV.

Table C1

Estimated Poles for Fifty (50) Runs of the Noisy Signal

of Example 1

Method Used: Pencil-of-Functions

Pole 1		Pole 2		Pole 3	
Real	Imag.	Real	Imag.	Real	Imag.
0.8248	0.0	0.8780	0.2962	0.8780	-0.2962
0.8407	0.0	0.8741	0.3091	0.8741	-0.3091
0.8323	0.0	0.8627	0.3081	0.8627	-0.3081
0.8530	0.0	0.8750	0.3196	0.8750	-0.3196
0.8359	0.0	0.8849	0.3116	0.8849	-0.3116
0.8504	0.0	0.8691	0.3192	0.8691	-0.3192
0.8501	0.0	0.8879	0.3150	0.8879	-0.3150
0.8231	0.0	0.8632	0.3054	0.8632	-0.3054
0.8347	0.0	0.8717	0.2871	0.8717	-0.2871
0.8422	0.0	0.8711	0.3132	0.8711	-0.3132
0.8471	0.0	0.8606	0.3137	0.8606	-0.3137
0.8312	0.0	0.8783	0.3081	0.8783	-0.3081
0.8301	0.0	0.8951	0.2904	0.8951	-0.2904
0.8428	0.0	0.8882	0.3058	0.8882	-0.3058
0.8393	0.0	0.8726	0.3062	0.8726	-0.3062
0.8414	0.0	0.8661	0.3074	0.8661	-0.3074
0.8489	0.0	0.8840	0.3179	0.8840	-0.3179
0.8518	0.0	0.8826	0.3040	0.8826	-0.3040
0.8279	0.0	0.8666	0.3143	0.8666	-0.3143
0.8312	0.0	0.8885	0.2889	0.8885	-0.2889
0.8332	0.0	0.8722	0.3273	0.8722	-0.3273
0.8561	0.0	0.8835	0.3068	0.8835	-0.3068
0.8411	0.0	0.8907	0.3069	0.8907	-0.3069
0.8538	0.0	0.8691	0.3161	0.8691	-0.3161
0.8437	0.0	0.8823	0.3099	0.8823	-0.3099
0.8288	0.0	0.8787	0.2983	0.8787	-0.2983
0.8370	0.0	0.8760	0.3014	0.8760	-0.3014
0.8490	0.0	0.8764	0.3193	0.8764	-0.3193
0.8589	0.0	0.8672	0.3212	0.8672	-0.3212
0.8128	0.0	0.8719	0.2889	0.8719	-0.2889
0.8368	0.0	0.8839	0.3128	0.8839	-0.3128
0.8392	0.0	0.8853	0.3134	0.8853	-0.3134
0.8547	0.0	0.8830	0.3170	0.8830	-0.3170
0.8448	0.0	0.8982	0.3153	0.8982	-0.3153
0.8440	0.0	0.8764	0.3181	0.8764	-0.3181
0.8348	0.0	0.8834	0.3094	0.8834	-0.3094
0.8365	0.0	0.8774	0.2962	0.8774	-0.2962
0.8431	0.0	0.8929	0.3083	0.8929	-0.3083
0.8517	0.0	0.8659	0.3316	0.8659	-0.3316
0.8509	0.0	0.8686	0.3219	0.8686	-0.3219
0.8497	0.0	0.8852	0.3030	0.8852	-0.3030
0.8487	0.0	0.8806	0.3044	0.8806	-0.3044
0.7981	0.0	0.8851	0.2671	0.8851	-0.2671
0.8203	0.0	0.8794	0.3053	0.8794	-0.3053
0.8495	0.0	0.8870	0.3099	0.8870	-0.3099
0.8234	0.0	0.8760	0.3096	0.8760	-0.3096
0.8485	0.0	0.8750	0.3153	0.8750	-0.3153
0.8412	0.0	0.8956	0.2985	0.8956	-0.2985
0.8516	0.0	0.8758	0.3249	0.8758	-0.3249
0.8521	0.0	0.8850	0.3106	0.8850	-0.3106

Table C2

Estimated Poles for Fifty (50) Runs of the Noisy Signal
of Example 1

Method Used: Pencil-of-Functions With Noise Correction

Pole 1		Pole 2		Pole 3	
Real	Imag.	Real	Imag.	Real	Imag.
0.8287	0.0	0.9223	0.2738	0.9223	-0.2738
0.8485	0.0	0.9100	0.2944	0.9100	-0.2944
0.8421	0.0	0.8972	0.2977	0.8972	-0.2977
0.8613	0.0	0.9121	0.3043	0.9121	-0.3043
0.8405	0.0	0.9165	0.2957	0.9165	-0.2957
0.8582	0.0	0.9105	0.3002	0.9105	-0.3002
0.8554	0.0	0.9167	0.3001	0.9167	-0.3001
0.8331	0.0	0.9123	0.2906	0.9123	-0.2906
0.8423	0.0	0.9142	0.2686	0.9142	-0.2686
0.8502	0.0	0.9072	0.2985	0.9072	-0.2985
0.8550	0.0	0.8944	0.2971	0.8944	-0.2971
0.8390	0.0	0.9298	0.2877	0.9298	-0.2877
0.8362	0.0	0.9356	0.2713	0.9356	-0.2713
0.8485	0.0	0.9119	0.2957	0.9119	-0.2957
0.8459	0.0	0.9091	0.2890	0.9091	-0.2890
0.8470	0.0	0.9004	0.2882	0.9004	-0.2882
0.8562	0.0	0.9216	0.3014	0.9216	-0.3014
0.8575	0.0	0.9162	0.2855	0.9162	-0.2855
0.8367	0.0	0.9067	0.3014	0.9067	-0.3014
0.8353	0.0	0.9273	0.2681	0.9273	-0.2681
0.8393	0.0	0.9042	0.3152	0.9042	-0.3152
0.8628	0.0	0.9180	0.2888	0.9180	-0.2888
0.8467	0.0	0.9258	0.2895	0.9258	-0.2895
0.8619	0.0	0.9042	0.2999	0.9042	-0.2999
0.8488	0.0	0.9078	0.2965	0.9078	-0.2965
0.8351	0.0	0.9145	0.2827	0.9145	-0.2827
0.8429	0.0	0.9161	0.2816	0.9161	-0.2816
0.8549	0.0	0.9092	0.3025	0.9092	-0.3025
0.8659	0.0	0.9076	0.2997	0.9076	-0.2997
0.8209	0.0	0.9134	0.2748	0.9134	-0.2748
0.8431	0.0	0.9234	0.2953	0.9234	-0.2953
0.8460	0.0	0.9299	0.2934	0.9299	-0.2934
0.8629	0.0	0.9230	0.2995	0.9230	-0.2995
0.8495	0.0	0.9236	0.3024	0.9236	-0.3024
0.8484	0.0	0.9097	0.2995	0.9097	-0.2995
0.8405	0.0	0.9146	0.2953	0.9146	-0.2953
0.8432	0.0	0.9124	0.2804	0.9124	-0.2804
0.8474	0.0	0.9208	0.2929	0.9208	-0.2929
0.8589	0.0	0.8956	0.3177	0.8956	-0.3177
0.8584	0.0	0.9091	0.3029	0.9091	-0.3029
0.8550	0.0	0.9104	0.2897	0.9104	-0.2897
0.8542	0.0	0.9126	0.2870	0.9126	-0.2870
0.8046	0.0	0.9260	0.2519	0.9260	-0.2519
0.8272	0.0	0.9106	0.2946	0.9106	-0.2946
0.8543	0.0	0.9198	0.2906	0.9198	-0.2906
0.8304	0.0	0.9115	0.2966	0.9115	-0.2966
0.8555	0.0	0.9124	0.2975	0.9124	-0.2975
0.8465	0.0	0.9223	0.2854	0.9223	-0.2854
0.8587	0.0	0.9082	0.3112	0.9082	-0.3112
0.8590	0.0	0.9232	0.2917	0.9232	-0.2917

Table C3

Estimated Poles for Fifty (50) Runs of the Noisy Signal of Example 1

Method Used: Prony (or All-Pole Covariance)

Pole 1		Pole 2		Pole 3	
Real	Imag.	Real	Imag.	Real	Imag.
0.7884	0.0	−0.4171	0.0	0.0552	0.0
0.8184	0.0	−0.1840	0.0426	−0.1840	−0.0426
0.8087	0.0	−0.1797	0.2130	−0.1797	−0.2130
0.8244	0.0	−0.1486	0.2706	−0.1486	−0.2706
0.7762	0.0	−0.5476	0.0	0.2528	0.0
0.8182	0.0	−0.2194	0.0	−0.1087	0.0
0.7642	0.0	−0.2156	0.0	0.1252	0.0
0.7969	0.0	−0.4744	0.0	0.0461	0.0
0.7979	0.0	−0.1186	0.4021	−0.1186	−0.4021
0.7982	0.0	−0.4296	0.0	0.0885	0.0
0.7813	0.0	−0.4704	0.0	0.1590	0.0
0.8070	0.0	−0.1618	0.4163	−0.1618	−0.4163
0.7954	0.0	−0.2521	0.0	−0.0181	0.0
0.7574	0.0	−0.5090	0.0	0.4022	0.0
0.7788	0.0	−0.4379	0.0	0.1270	0.0
0.7468	0.0	−0.3464	0.0	0.1156	0.0
0.2613	0.0	0.7826	0.0	−0.6150	0.0
0.3870	0.0	0.7759	0.0	−0.6023	0.0
0.7692	0.0	−0.1222	0.3752	−0.1222	−0.3752
0.3543	0.0	−0.6700	0.0	0.7706	0.0
0.7547	0.0	−0.2677	0.0	0.0116	0.0
0.8529	0.0	−0.3008	0.0	−0.0255	0.0
0.7767	0.0	−0.2661	0.0	0.1377	0.0
0.7962	0.0	−0.3719	0.0	0.1047	0.0
0.7540	0.0	−0.3987	0.0	0.3012	0.0
0.7944	0.0	−0.2237	0.0	−0.0779	0.0
0.8068	0.0	−0.3446	0.0	−0.0091	0.0
0.7731	0.0	−0.2018	0.0	−0.0566	0.0
0.7827	0.0	−0.5571	0.0	0.2154	0.0
0.2098	0.0	0.7761	0.0	−0.6244	0.0
0.7999	0.0	−0.1537	0.3256	−0.1537	−0.3256
0.7866	0.0	−0.2558	0.0	0.0240	0.0
0.8365	0.0	−0.2889	0.0	−0.0384	0.0
0.7855	0.0	−0.3534	0.0	0.2192	0.0
0.7676	0.0	−0.1320	0.1508	−0.1320	−0.1508
0.7902	0.0	−0.4645	0.0	0.1714	0.0
0.7805	0.0	−0.4361	0.0	0.1347	0.0
0.7629	0.0	−0.3311	0.0	0.1737	0.0
0.3584	0.0	0.7315	0.0	−0.5916	0.0
0.7651	0.0	−0.4343	0.0	0.1240	0.0
0.7965	0.0	−0.3931	0.0	0.2374	0.0
0.7991	0.0	−0.3932	0.0	0.1353	0.0
0.7835	0.0	−0.1486	0.2030	−0.1486	−0.2030
0.7904	0.0	−0.4655	0.0	0.1185	0.0
0.8144	0.0	−0.0909	0.2120	−0.0909	−0.2120
0.7876	0.0	−0.0753	0.3471	−0.0753	−0.3471
0.7819	0.0	−0.1089	0.0279	−0.1089	−0.0279
0.8052	0.0	−0.3982	0.0	0.1758	0.0
0.7904	0.0	−0.1299	0.2539	−0.1299	−0.2539
0.8098	0.0	−0.1021	0.3111	−0.1021	−0.3111

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APPENDIX D

Computer Output of 'Analysis of Scatterer Response by Pencil-of-Functions Method'

A. SCATTERER RESP. ANALYSIS (BIAS NOT REMOVED, OR ESTIMATED)
245 011 8 0+2 0 245 +00.00000 +0.100000 +000.0000
SCAT-NC 0-7/8
PR IZTS IREM ISPN IFIX NFIX IBIAS IBO MNPT GI DF (NOISE CORREC.)
1-0 1-1 1 1+0 0 000 +00.80000 +00.0011

```
SYNTHETIC VECTOR, AND SQRT(Y11)
1.0000   2.0168   1.9186   1.0954   .40462   .98053D-01  .15126D-01  .13519D-02  .53399D-04  .10759D-23
ESTIMATED PARAMETER VECTOR
1.00000  2.01683  1.91855  1.07541  .404617  .980527D-01 .151258D-01 .135190D-02 .53987D-04
ESTIMATED PARAMETER VECTOR
1.00000  24.6887  -7.47897  -46.9972  56.4378  -43.7880  21.4367  -6.05436  .755240

NOISY GRAM MATRIX (DET= .593911D-52)
2.41680   .939289   .303130   .781030D-01  .138023D-01  .732192D-03 -.384471D-05  .885036D-03
 .939289   .449286   .178890   .610722D-01  .179564D-01  .462856D-02  .123508D-02  .377982D-03
 .303130   .178890   .839109D-01 .338948D-01  .122064D-01  .404420D-02  .131616D-02  .225389D-03
 .781030D-01 .610722D-01 .338948D-01 .158360D-01  .661243D-02  .255830D-02  .953665D-03  .150676D-03
 .138023D-01 .179564D-01 .122064D-01 .661243D-02  .315004D-02  .137967D-02  .573452D-03  .969528D-04
 .732192D-03 .462856D-02 .404420D-02 .255830D-02  .137967D-02  .673039D-03  .306954D-03  .578124D-04
-.384471D-05 .123508D-02 .131616D-02 .953665D-03  .573452D-03  .306954D-03  .151677D-03  .320391D-04
-.696325D-03 .537959D-03 .481730D-03 .362633D-03  .233701D-03  .134298D-03  .709181D-04   .167030D-04
 .885036D-03 .377982D-03 .225389D-03 .150676D-03  .969528D-04  .578124D-04  .320391D-04   .830129D-05

J.I GDET.SUMDET.SII 1 0  0.59D-52  0.34D+05  0.30D-04
GDET.DFAC.THR:  0.59D-52  0.11D-02  0.65D-55
J.I SI.CDET 1 0  0.59D-52  0.30D-04  0.31D-53
J.I GDET.SUMDET.SII 2 0  0.31D-53  0.59D+06  0.17D-05
J.I SI.CDET 2 0  0.31D-53  0.17D-05  0.10D-55
ESTIMATED NOISE VAR= .31573D-04

SYNTHETIC VECTOR, AND SQRT(Y11)
1.0000   1.9452   1.7711   .96248   .33706  .77276D-01  .11273D-01  .35727D-03  .10269D-23
ESTIMATED PARAMETER VECTOR
1.00000  1.94522  1.77107  .962481  .337057  .772763D-01 .112734D-01 .953770D-03  .357273D-04
ESTIMATED PARAMETER VECTOR
1.00000  25.2414  -7.56847  -48.4654  58.6132  -45.7300  22.4815  -6.36797  .795745

GRST MATRIX (DET= .100805D-55)
2.41602   .939211   .303122   .781022D-01  .138022D-01  .732184D-03 -.384549D-05  .885036D-03
 .939211   .449264   .178884   .610706D-01  .179559D-01  .462844D-02  .123505D-02  .377980D-03
 .303122   .178884   .839082D-01 .338938D-01  .122060D-01  .404408D-02  .131612D-02  .225385D-03
 .781022D-01 .610706D-01 .338938D-01 .158355D-01  .661222D-02  .255822D-02  .953636D-03  .150672D-03
 .138022D-01 .179559D-01 .122060D-01 .661222D-02  .314994D-02  .137963D-02  .573434D-03  .969502D-04
 .732184D-03 .462844D-02 .404408D-02 .255822D-02  .137963D-02  .673017D-03  .306945D-03  .578108D-04
-.384549D-05 .123505D-02 .131612D-02 .953636D-03  .573434D-03  .306945D-03  .151672D-03  .320381D-04
-.696325D-03 .537950D-03 .481717D-03 .362623D-03  .233694D-03  .134293D-03  .709159D-04   .167025D-04
 .885036D-03 .377980D-03 .225385D-03 .150672D-03  .969502D-04  .578108D-04  .320381D-04   .830105D-05
```

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EST TF B(Z)/A(Z)   (FMAX= 0.17050D+00)

```
  1.000000    -7.358474    25.241444   -48.465441    98.613231
-45.730035    22.481500    -6.367773     0.795745    -0.662435
  0.0          0.070030    -0.338930     0.649220
  0.371143    -0.097361     0.006332     0.0
```

SS SIGNAL= .702536D-01   SS MODEL= .697222D-01
SS ERROR= .627433D-03MAX= .128043D-01 RATIO= .693097D-02
```
.9961      .9961      .9800      .9800      .9447      .9447      .9447      .8655
.8655
-.1696D-01 -.1696D-01  .1322      .1322     -.1322     -.1784     -.1784
-.3762      .3762
```

8-POLES
	SR	SI	SNAG	FR	IZTB= 0
1	-1.9461	-8.7184	8.9329	1.4217	
2	-1.9461	8.7184	8.9329	1.4217	
3	-5.7185	-68.6328	68.8706	10.9611	
4	-5.7185	68.6328	68.8706	10.9611	
5	-20.1755	-95.5826	97.6887	15.5476	
6	-20.1755	95.5826	97.6887	15.5476	
7	-30.6498	-210.3752	212.5962	33.8357	
8	-30.6498	210.3752	212.5962	33.8357	

NUMERATOR CONSTANTS OF FACTORIZED H(S)
```
-0.16464D+01 -0.91174D+00 -0.16464D+01 0.91174D+00 -0.40361D+01 0.57694D+00 -0.40361D+01 -0.57694D+00
 0.43706D+01 -0.11887D+01  0.13120D+01 -0.61173D+00  0.13120D+01 0.61173D+00  0.43706D+01  0.11887D+01
```

8-DOMAIN DENOMINATOR
```
0.345698D-01   0.100484D-02   0.120935D-03   0.553745D-02   0.109005D-03   0.269117D-02
0.242765D-02   0.228476D-01   0.100000D+01
```

8-DOMAIN NUMERATOR
```
 0.177477D-02  -0.280087D-03  -0.242715D-03  -0.595324D-03  -0.191955D-03  -0.694545D-02
-0.409756D+01   0.619393D-11   0.0
```

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B SCAT 0-7/8

PR IZTS IREN ISPN IFIX NFIX IBIAS IBO NNPT QI DF (UNCORREC.)
1-0 1-1-1-1-0 0 000 +00.80000

SYNTHETIC VECTOR, AND SQRT(Y11)
1.0000   2.0168   1.9186   1.0954   .40462   .98053D-01   .15126D-01   .13519D-02   .53399D-04   .10759D-23

ESTIMATED PARAMETER VECTOR
1.00000   2.01683   1.91855   1.09541   .404617   .980527D-01   .151258D-01   .135190D-02   .533997D-04

ESTIMATED PARAMETER VECTOR
1.00000   -7.47897   24.6887   -46.9972   56.4378   -43.7880   21.4367   -6.05436   .755240

NOISY GRAM MATRIX (DET= .593911D-52)

2.41680	.939289	.303130	.781030D-01	.138023D-01	.732192D-03	-.384471D-05	.696325D-03	.885036D-03
.939289	.449286	.178890	.610722D-01	.179564D-01	.462856D-02	.123508D-02	.537959D-03	.377982D-03
.303130	.178890	.839109D-01	.338948D-01	.122064D-01	.404420D-02	.131616D-02	.481730D-03	.223389D-03
.781030D-01	.610722D-01	.338948D-01	.158360D-01	.661243D-02	.255830D-02	.953665D-03	.362633D-03	.150676D-03
.138023D-01	.179564D-01	.122064D-01	.661243D-02	.315004D-02	.137967D-02	.573452D-03	.237301D-03	.969528D-04
.732192D-03	.462856D-02	.404420D-02	.255830D-02	.137967D-02	.673039D-03	.306954D-03	.134298D-03	.578124D-04
-.384471D-05	.123508D-02	.131616D-02	.953665D-03	.573452D-03	.306954D-03	.151677D-03	.709181D-04	.320391D-04
.696325D-03	.537959D-03	.481730D-03	.362633D-03	.237301D-03	.134298D-03	.709181D-04	.351854D-04	.167030D-04
.885036D-03	.377982D-03	.225389D-03	.150676D-03	.969528D-04	.578124D-04	.320391D-04	.167030D-04	.830129D-05

EBT TF B(2)/A(2)   (FMAX= 0.17050D+00)
1.000000   -7.478967   24.688702   -46.997176   56.437830
-43.787984   21.436719   -6.054364   0.755240
0.0   -0.040154   0.340988   -1.085657   1.748931
-1.540066   0.712035   -0.136076   0.0

SS SIGNAL= .702536D-01 SS MODEL= .594162D-01
SS ERROR= .107180D-01IMAX= .424738D-01 RATIO= .152561
.9888   .9888   .9708   .9708   .9149   .9149
.8649   .8649
-.2257D-01 -.2257D-01   .1300   -.1300   .2073   .2073   -.2073
-.4078   .4078

8-POLES   IZTS= 0

	SR	SI	SMAG	FR
1	-5.6153	-11.6852	12.9644	2.0634
2	-5.6153	11.6852	12.9644	2.0634
3	-10.6245	-68.1463	68.9696	10.9768
4	-10.6245	68.1463	68.9696	10.9768
5	-32.7052	-114.0767	118.6723	18.8873
6	-32.7052	114.0767	118.6723	18.8873
7	-22.9193	-225.5644	226.7259	36.0845
8	-22.9193	225.5644	226.7259	36.0845

NUMERATOR CONSTANTS OF FACTORIZED H(S)
-0.24899D+01-0.26692D+00   -0.24899D+01 0.26692D+00   -0.66104D+01 0.60482D+00   -0.66104D+01-0.60482D+00
0.71258D+01 0.38857D+01   0.71258D+01-0.38857D+01   0.19745D+01-0.29419D+00   0.19745D+01 0.29419D+00   0.71258D+01+0.38857D+01-0.38857D+01

B-DOMAIN DENOMINATOR
0.122564D+02   0.482076D+02   0.223224D+03   0.136994D+03   0.176789D+03   0.511391D+02
0.294735D+02   0.280720D+01   0.100000D+01

B-DOMAIN NUMERATOR
-0.475182D+02   -0.923385D+03   -0.170571D+03   -0.148234D+04   0.311190D+02   -0.111115D+03
0.108374D+02   0.564819D-11   0.0

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